# Copula Generation: Theory and Visualization


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# 1. Introduction

This document is dedicated to the topic of copula generation methods, which are essential tools for extending or modifying the structure of existing copulas. The material focuses on three such methods: rotation, the Khoudraji device, and mixtures of copulas.

To illustrate these methods, the document uses four well-known bivariate copulas: the Gaussian and t-Student (elliptical copulas), and the Clayton and Gumbel (Archimedean copulas). These copulas are not the primary object of study, but they serve as essential building blocks for understanding how the generation methods are implemented.

While some definitions and theorems are presented in general $d$-dimensional form, the document primarily concentrates on the formulas and construction logic behind each generation method in the bivariate case.

Additionally, the document provides analytic expressions for dependence coefficients, including Kendall's Tau, Spearman's Rho, and upper and lower tail dependence coefficients, both in general form and as derived for the selected copulas. These coefficients help quantify the strength and nature of dependence inherent in each copula, providing important theoretical context for the generation processes described.

# 2. $d$-Dimensional Copula Function

A $d$-dimensional ($d \geq 2$) copula function is a multivariate cumulative distribution function (CDF):

$$C(u_1, u_2, \dots, u_d)$$

where $u_i \in [0,1]$ for all $i = 1, \dots, d$, which satisfies (most simple properties):

- $C: [0,1]^d \to [0,1]$.
- $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$.
- $C(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_d) = 0$ (at least one 0).
- $C(1, \dots, 1) = 1$.
- $d$-increasingness: The volume of any rectangle in $[0,1]^d$ is non-negative.

# 3. Sklar's Theorem

For any joint CDF $F(x_1, x_2, \dots, x_d)$ with marginals $F_1(x_1), F_2(x_2), \dots, F_d(x_d)$, there exists a copula $C$ such that:

$$F(x_1, x_2, \dots, x_d) = C\big(F_1(x_1), F_2(x_2), \dots, F_d(x_d)\big).$$

Conversely, for any copula $C$ and marginals, the following holds:

$$C(u_1, \dots, u_d) = F\Big(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\Big)$$

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where $u_i \in [0,1]$. Each $u_i$ represents a realization of a uniformly distributed random variable $U_i$ on $[0,1]$. Here, $F_i^{-1}$ denotes the generalized inverse (quantile function) of $F_i$. If the marginal distributions are continuous, the copula $C$ is unique.

# 4. CDF and PDF of a Bivariate Random Variable with Respect to the Bivariate Copula CDF and PDF

Relationship between the CDF of a bivariate random variable and the corresponding copula CDF:

$$F(x_1, x_2) = C\big(F_1(x_1), F_2(x_2)\big) = C(u_1, u_2)$$

where $F(x_1, x_2)$ is the joint CDF of a bivariate random variable, and $C(u_1, u_2)$ is the bivariate copula CDF.

Relationship between the PDF of a bivariate random variable and the corresponding bivariate copula PDF:

$$f(x_1, x_2) = c\big(F_1(x_1), F_2(x_2)\big) \cdot f_1(x_1) \cdot f_2(x_2) = c(u_1, u_2) \cdot f_1(x_1) \cdot f_2(x_2)$$

where $f(x_1, x_2)$ is the joint PDF of a bivariate random variable, and $c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}$ is the bivariate copula PDF.

# 5. Definitions of Selected Bivariate Copulas

## 5.1 Bivariate Gaussian Copula (Elliptical Copula)

$$C(u_1, u_2; \rho) = \Phi_\rho\big(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\big)$$

where:

- $\Phi^{-1}$: The inverse of the univariate standard normal cumulative distribution function $\Phi$.
- $\Phi_\rho$: The joint two-variate standard normal cumulative distribution function and correlation coefficient $\rho \in [-1,1]$.

## 5.2 Bivariate Student's $t$ Copula (Elliptical Copula)

$$C(u_1, u_2; \rho, \nu) = T_{\rho,\nu}\big(T_\nu^{-1}(u_1), T_\nu^{-1}(u_2)\big)$$

where:

- $T_\nu^{-1}$: The inverse of the univariate standard Student's $t$ cumulative distribution function $T_\nu$ and $\nu > 0$ degrees of freedom (df).

- $T_{\rho,\nu}$: The joint two-variate standard Student's $t$ cumulative distribution function with correlation coefficient $\rho \in [-1,1]$ and $\nu > 0$ degrees of freedom (df).

### 5.3 Bivariate Clayton Copula (Archimedean Copula)

$$C(u_1, u_2; \theta) = \left({u_1}^{-\theta} + {u_2}^{-\theta} - 1\right)^{-\frac{1}{\theta}}$$

where $\theta \in (0, +\infty)$.

### 5.4 Bivariate Gumbel Copula (Archimedean Copula)

$$C(u_1, u_2; \theta) = \exp\left[-\left((-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right)^{\frac{1}{\theta}}\right]$$

where $\theta \in [1, +\infty)$.

### 5.5 Independence Copula

$$C(u_1, u_2) = u_1 \cdot u_2$$

## 6. Coefficients of Dependence

### 6.1 General Formulas

#### 6.1.1 Kendall's Tau

$$\tau = 4 \iint_{[0,1]^2} C(u_1, u_2) dC(u_1, u_2) - 1 = 4 \cdot E\big(C(U_1, U_2)\big) - 1$$

where $\tau \in [-1,1]$. Here, $E\big(C(U_1, U_2)\big)$ represents the expected value of the copula's cumulative distribution function (CDF), which depends on two random variables $U_1$ and $U_2$, that are uniformly distributed on the interval $[0,1]$.

#### 6.1.2 Spearman's Rho

$$\rho_S = 12 \iint_{[0,1]^2} u_1 u_2 dC(u_1, u_2) - 3 = 12 \cdot E(U_1 U_2) - 3$$

where $\rho_S \in [-1,1]$, and $E(U_1 U_2)$ denotes the expected value of the product of two random variables, $U_1$ and $U_2$, that are uniformly distributed on the interval $[0,1]$.

### 6.1.3 Lower Tail Dependence

$$\begin{aligned}\lambda_L &= lim_{u\to 0^+} P(X_2 \leqslant F_2^{-1}(u) \mid X_1 \leqslant F_1^{-1}(u)) \\ &= lim_{u\to 0^+} \frac{P(X_2 \leqslant F_2^{-1}(u), X_1 \leqslant F_1^{-1}(u))}{P(X_1 \leqslant F_1^{-1}(u))} \\ &= lim_{u\to 0^+} \frac{C(u,u)}{C(u,1)} = lim_{u\to 0^+} \frac{C(u,u)}{u}\end{aligned}$$

where $\lambda_L$ is the lower tail dependence coefficient.

### 6.1.4 Upper Tail Dependence

$$\begin{aligned}\lambda_U &= lim_{u\to 1^-} P(X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)) \\ &= lim_{u\to 1^-} \frac{P(X_2 > F_2^{-1}(u), X_1 > F_1^{-1}(u))}{P(X_1 > F_1^{-1}(u))} \\ &= lim_{u\to 1^-} \frac{1 + C(u,u) - 2u}{1 - C(u,1)} \\ &= lim_{u\to 1^-} \frac{1 + C(u,u) - 2u}{1 - u}\end{aligned}$$

where $\lambda_U$ is the upper tail dependence coefficient.

## 6.2 Copula Specific Formulas

### 6.2.1 Copula specific Kendall's Tau and Spearman's Rho coefficients

| | Copula | Kendall's Tau | Spearman's Rho |
|---|---|---|---|
| ELLIPTICAL | Gaussian<br>$\rho \in [-1,1]$<br>$\tau \in [-1,1]$<br>$\rho_S \in [-1,1]$ | $\tau = \frac{2}{\pi}\arcsin(\rho)$ | $\rho_S = \frac{6}{\pi}\arcsin\left(\frac{\rho}{2}\right)$ |
| ELLIPTICAL | t-Student<br>$\rho \in [-1,1]$<br>$\tau \in [-1,1]$<br>$\rho_S \in [-1,1]$ | $\tau = \frac{2}{\pi}\arcsin(\rho)$ | It is not easy to exhibit $\rho_S$ |
| ARCHIMEDEAN | Clayton<br>$\theta \in (0,+\infty)$<br>$\tau \in (0,1)$<br>$\rho_S \in (0,1)$ | $\tau = \frac{\theta}{\theta+2}$ | |
| ARCHIMEDEAN | Gumbel<br>$\theta \in [1,+\infty)$<br>$\tau \in [0,1)$<br>$\rho_S \in [0,1)$ | $\tau = 1 - \frac{1}{\theta}$ | |

### 6.2.2 Copula specific Tail Dependencies

| | Copula | Lower Tail Dependence | Upper Tail Dependence |
|---|---|---|---|
| ELLIPTICAL | Gaussian | $\lambda_L = 0$ | $\lambda_U = 0$ |
| | t-Student<br>$\lambda_{L,U} \in (0,1]$ | $\lambda_L = 2T_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right)$ | $\lambda_U = 2T_{\nu+1}\left(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}}\right)$ |
| ARCHIMEDEAN | Clayton<br>$\lambda_L \in (0,1)$ | $\lambda_L = 2^{-1/\theta}$ | $\lambda_U = 0$ |
| | Gumbel<br>$\lambda_U \in [0,1)$ | $\lambda_L = 0$ | $\lambda_U = 2 - 2^{1/\theta}$ |

$T_{\nu+1}$ is the CDF of the t-Student distribution with $df = \nu + 1$

# 7. Copula Generation

## 7.1 Rotation

General rotation formula for a point in $\mathbb{R}^2$:

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = R \cdot \begin{bmatrix} x \\ y \end{bmatrix} - (R - I) \cdot \begin{bmatrix} a \\ b \end{bmatrix}$$

where: $\mathrm{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is a rotation matrix, $\alpha$ is a rotation angle, $\begin{bmatrix} x \\ y \end{bmatrix}$ is a original point, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the $2 \times 2$ identity matrix, and $\begin{bmatrix} a \\ b \end{bmatrix}$ is a rotation center.

In the context of bivariate copulas, $\alpha$ should be a multiple of 90°, $a = b = 0.5$, and $x\ and\ y$ are within the interval $[0,1]$.

Formulas for rotated copula PDFs:

- $c_{0^\circ}(u_1, u_2) = c(u_1, u_2)$
- $c_{90^\circ}(u_1, u_2) = c(1 - u_2, u_1)$
- $c_{180^\circ}(u_1, u_2) = c(1 - u_1, 1 - u_2)$
- $c_{270^\circ}(u_1, u_2) = c(u_2, 1 - u_1)$
- $c_{360^\circ}(u_1, u_2) = c(u_1, u_2)$

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## 7.2 Khoudraji Copula (Device)

Formula for the Khoudraji copula (device). Dimension $d \geqslant 2$ ($\boldsymbol{u} \in [0,1]^d$):

$$C_{KHO}(\boldsymbol{u}) = C_1(\boldsymbol{u}^{1-\boldsymbol{s}}) \cdot C_2(\boldsymbol{u}^{\boldsymbol{s}})$$

where: $C_1$ and $C_2$ are arbitrary copula CDFs, $\boldsymbol{s} = (s_1, \ldots, s_d)$ is a vector of shape parameters with $s_i \in [0,1]$, and $\boldsymbol{u}^{\boldsymbol{p}} = \left(u_1^{p_1}, u_2^{p_2}, \ldots, u_d^{p_d}\right)$.

Formula for the Khoudraji copula (device). Dimension $d = 2$ ($\boldsymbol{u} \in [0,1]^2$):

$$C_{KHO}(u_1, u_2) \quad = C_1\left(u_1^{1-s_1}, u_2^{1-s_2}\right) \cdot C_2\left(u_1^{s_1}, u_2^{s_2}\right)$$

where $s_1$ and $s_2$ are shape parameters in $[0,1]$.

## 7.3 Mixture of Copulas

Formula for the mixture of copulas. Dimension $d \geqslant 2$ ($\boldsymbol{u} \in [0,1]^d$):

$$C_{MIX}(\boldsymbol{u}) = \sum_{k=1}^{m} w_k \cdot C_k(\boldsymbol{u})$$

where: $C_1 \ldots, C_m$ are arbitrary copulas, $w_k \in [0,1]$, $\sum_{k=1}^{m} w_k = 1$, and $m \geq 2$.

Formula for the mixture of copulas. Dimension $d = 2$ ($\boldsymbol{u} \in [0,1]^{\boldsymbol{2}}$):

$$C_{MIX}(u_1, u_2) \quad = w_1 \cdot C_1(u_1, u_2) + w_2 \cdot C_2(u_1, u_2)$$

where $w_1 \in [0,1]$ and $w_2 = 1 - w_1$.

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